THESIS

An Analysis of Lock Foundation Design

by

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This is to certify that the undersigned have read and recommended to the Committee on Graduate Studies for acceptance, a thesis submitted by Elio D'Appolonia, B.Sc. entitled:

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Professor

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THESIS

AN ANALYSIS OF LOCK FOUNDATION DESIGN

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The Analysis of Lock Foundation Design

Introduction

The design of structures, such as locks, footings and culverts resting on a semi-infinite body, is dependent on the analysis of the contact pressure between the slab and the foundation. For footings and culverts the assumption normally made is that the contact pressure is uniform over the area of the slab. In the case of locks and similar strip foundations it is assumed that the deflection is proportional to the pressure. This assumption gives rise to the hypothetical concept of a modulus of the foundation. By means of this simplifying assumption the contact pressure can then be determined from a solution of the differential equation (1) subject to the particular boundry conditions.

$$D\frac{d^4\omega}{dx^4} = q(x) - k\omega \qquad \text{---}(1)$$

$q(x)$ is the load on the slab, and D the flexural rigidity. The modulus of the foundation k is defined by (2) as

$$k = \frac{p}{\omega} \qquad \text{---}(2)$$

in which p and ω are the contact pressure and the deflection respectively for any given point of the slab.

The assumption of a constant modulus of the foundation is, in general incompatible with the mechanical behavior of solids and soils. To have reality agree with theory it would be necessary that the supporting media be made of identical but individual springs each of which could function independently of the other. In other words the hypothesis implies that the deflection is independent of the loaded area and that a load can be the cause only of the deflection underneath the loaded area. Computations based on the use of equation (1) are at the most crude estimations.

The theory of the bending of a bar on an elastic foundation in which the hypothesis of the modulus of the foundation is adopted dates back to 1867.. In 1888 [1] the method was used in the design of railways. Mathematical expressions [2] developed from (1) for the case of concentrated loads are applied to the design of highway and runway slabs. For such applications of equation (1) it has been shown that the computed behavior of the slab checks, with fair agreement, field measurements. The k value for such designs is computed from field tests [3] or from theoretical results [4].

For foundation slabs subject to uniform or varying load, overturning moments or concentrated loads at the edges of the strip such as occur in locks, there appears to be no obvious justification for the vigorous application of a theoretical analysis based on the hypothetical k value. Nevertheless in the literature there can be found many examples[5] of this method of analysis. The modulus of the foundation is generally determined from empirical expressions whose limitations or restrictions are meagrely described.

The factors which affect the variation of the contact pressure are the size, shape, elastic properties and flexural rigidity of the slab, the elastic properties of the soil, and the character of the superimposed load. Whenever equation (1) is to be adopted for a preliminary analysis of the contact pressure beneath a slab, a good approximation for a k value could be made from the proper selection of pressure and deflection values as determined by the theory of elasticity.

[1] Refer to the list of references in the bibliography

An average k value secured between the limits of a perfectly flexible and absolutely rigid slab would be commendable. Considering a few examples we have, for the case of the perfectly flexible circular slab subject to a uniform load,[6a] that the modulus of the foundation is given by

$$k_f = \left[\frac{p}{w}\right]_f = \frac{\pi E}{4a(1-\nu^2)\,E(k)} \qquad \text{--- (3)}$$

where $E(k)$ is the complete elliptic integral of the second kind, a, ν and E are the radius, Poisson's ratio, and the modulus of elasticity respectively for the slab. It can be seen that k depends on the size of the slab and varies from a value of

$$\left[k_f\right]_0 = \frac{E}{2a(1-\nu^2)} \qquad \text{--- (4)}$$

at the centre to a value of

$$\left[k_f\right]_a = \frac{\pi E}{4a(1-\nu^2)} \qquad \text{---- (5)}$$

at the edge. On the basis of an average slab deflection the value is

$$\left[k_f\right]_{av.} = \frac{3}{16}\,\frac{\pi E}{a(1-\nu^2)} \qquad \text{--- (6)}$$

In a similar manner for an absolutely rigid circular plate the modulus of foundation[6b] is given as

$$k_r = \left[\frac{p}{w}\right]_r = \frac{E}{\pi(1-\nu^2)\sqrt{a^2-r^2}} \qquad \text{--- (7)}$$

which gives a minimum value at the centre of

$$\left[k_r\right]_0 = \frac{E}{\pi a(1-\nu^2)} \qquad \text{---- (8)}$$

to infinity at the edge.

For a flexible square plate the k value computed from an average deflection[6a] is

$$k_{av.} = \frac{0.526\ E}{a(1-\nu^2)} \qquad --- (9)$$

and for rectangular plates

$$k_{av.} = \frac{E}{m\sqrt{A}\ (1-\nu^2)} \qquad --- (10)$$

where a is half the side of the square plate and A the area of the rectangle. Values of m for various ratios of the sides of the rectangle are given in reference 6a page 338. In the foregoing computations it has been assumed that both the slab and the foundation are elastic isotropic materials.

Object

In the first part of the work to follow, consideration will be given to the two-dimensional problem of an elastic strip resting on an elastic isotropic foundation. The solutions will be carried out for concentrated loads and overturning moments at the edges of the strip. The method of analysis will be similar to that given by Borowicka[7] for the case of the uniformly loaded slab. The solution takes into consideration the physical properties of the slab and the foundation. No simplifying assumption that the deflection is proportional to the pressure is made.

The second part will include a discussion and criticism of the validity of Borowicka's method of analysis for the cases of symmetrical but discontinuous loadings.

In the third part, consideration will be given to the inverse problems. That is, the equation for the foundation reaction is assumed and the deflection is then computed from Boussinesq's equation for a semi-infinite loaded mass. Substitution of this computed deflection into the

slab equation (1) yields the expression necessary to determine the superimposed load on the slab. Examples showing the determination of the contact pressure from assumed deflection curves will also be given.

Assumptions and Notation

The following assumptions are made with regard to the determination of the contact pressure and deflection for the slab and foundation;

(a) the strip or slab and the supporting foundation are homogenous elastic and isotropic materials.

(b) Hooke's Law and the principle of superposition are valid.

(c) that no tangential forces exist between the slab and the foundation.

The notation used is as follows;

E_s the modulus of elasticity of the slab

E_b the modulus of elasticity of the foundation assumed constant for the full depth of the soil

m_s Poisson's constant for the slab

m_b Poisson's constant for the foundation

a half the width of the slab

h the uniform thickness of the slab

q the uniform load on the slab

W the concentrated line load at the edge

M the overturning moment at the edge

w the deflection of the slab or foundation positive downwards

x, y Cartesion co-ordinates

ξ, ζ variables in the x, y plane

r distance from the load to point at which deflection is computed

$D = \dfrac{m_s^2 \; E_s \; h^3}{12(m_s^2 - 1)}$ the flexural rigidity of the slab

$N = \dfrac{m_b^2 \; E_b}{m_b^2 - 1}$ a constant from Boussinesq's equation

$K = \dfrac{2D}{N a^3}$ a constant

A_o, A_2, A_4 ------------)
)
B_o, B_2, B_4 ------------) coefficients
)
C_o, C_1, C_2 ------------)

α and β - constants

Theory

In the following method of analysis it is assumed that the contact pressure between the slab and the foundation can be represented by a Maclaurin series. With this assumption, the deflection of the slab is then determined from the slab equation:

$$\frac{d^4 \omega}{d x^4} = \frac{1}{D}\left[q(x) - p(x) \right] \quad \text{- - - (11)}$$

and the deflection of the foundation can be found from Boussinesq's expression [6b.]

$$\omega = \frac{1}{\pi} \; \frac{m_b^2 - 1}{m_b^2 \; E_b} \int_A \frac{p(\xi) \; d\xi \; d\zeta}{r} \quad \text{- - - - (12)}$$

For the deflections as determined by (11) and (12) to be compatible the coefficients of all like powers of x in the two equations have to be equal. In this manner an infinite set of equations involving the coefficients of the assumed power series for the contact pressure are obtained. The contact pressure can then be determined to any desired degree of accuracy from the solution of a finite number of the infinite set of equations.

Case 1

Concentrated Line Loads at the Edges of the Slab.

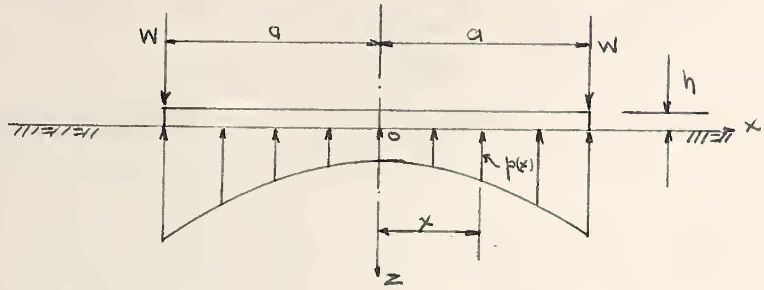

FIG 1.

For Case 1, shown in figure 1, we have for the boundry conditions of the upper surface of the foundation that

$$\left.\begin{array}{lll} x > a & \sigma_z = 0 & w \text{ any value} \\ x < a & \sigma_z = p(x) & \end{array}\right\} \quad \text{---(13)}$$

The boundry conditions for the slab are

$$\left.\begin{array}{ll} x = a & \dfrac{d^2w}{dx^2} = 0 \\ x = 0 & w = w_0 \; ; \quad \dfrac{dw}{dx} = 0 \; ; \quad \dfrac{d^3w}{dx^3} = 0 \end{array}\right\} \quad \text{--(14)}$$

The slab equation for case 1 is

$$\frac{d^4w}{dx^4} = -\frac{p(x)}{D} \qquad \text{---(15)}$$

The solution of (15) is given as

$$w = w_c + w_p$$

where w_c is the solution to the homogenous differential equation

$$\frac{d^4w}{dx^4} = 0 \qquad \text{----(16)}$$

and w_p is a particular integral in the non-homogenous differential equation (15).

We have from (16) that the complementary solution is

$$w_c = c_0 + c_1\left(\frac{x}{a}\right) + c_2\left(\frac{x}{a}\right)^2 + c_3\left(\frac{x}{a}\right)^3 \qquad \text{----(17)}$$

The particular integral is assumed to be a Maclaurin series of even powers with unknown coefficients, that is

$$w_p = \sum_{l=2}^{\infty} A_{2l}\left(\frac{x}{a}\right)^{2l} \qquad \text{----(18)}$$

Likewise it is assumed that the contact pressure $p(x)$ is represented by the power series

$$p(x) = \sum_{n=0}^{\infty} B_{2n} \left(\frac{x}{a}\right)^{2n} \qquad \text{- - - - (19)}$$

Substituting (19) in the slab equation (15) and integrating we have

$$w_p = -\frac{a^4}{D} \sum_{n=0}^{\infty} B_{2n} \frac{2n!}{(2n+4)!} \left(\frac{x}{a}\right)^{2n+4} \qquad \text{- - - (20)}$$

Comparing the coefficients between (18) and (20) we find

$$A_{2n+4} = -\frac{1}{D} a^4 B_{2n} \frac{2n!}{(2n+4)!}$$

The deflection of the slab can then be written as

$$w = c_0 + c_1\left(\frac{x}{a}\right) + c_2\left(\frac{x}{a}\right)^2 + c_3\left(\frac{x}{a}\right)^3 - \frac{a^4}{D} \sum_{n=0}^{\infty} \frac{2n!}{(2n+4)!} B_{2n}\left(\frac{x}{a}\right)^{2n+4}$$

From the boundry conditions of the slab we determine that the coefficient

$$c_0 = w_0$$

$$c_1 = c_3 = 0$$

$$c_2 = \frac{a^4}{2D} \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n+2)(2n+1)}$$

and the deflection for the slab becomes

$$w = w_0 + \frac{a^4}{2D} \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n+2)(2n+1)} \left(\frac{x}{a}\right)^2 - \frac{a^4}{D} \sum_{n=0}^{\infty} \frac{2n!}{(2n+4)!} B_{2n}\left(\frac{x}{a}\right)^{2n+4} \qquad \text{- - - (21)}$$

Considering now the deflection of the upper surface of the foundation which must conform to the deflection of the slab for a contact pressure of $p(x)$, we have that the deflection w at any point x due to load $p(\xi)$ is[6b]

$$w = 2\int_{-a}^{a}\int_{0}^{\infty} \frac{1}{\pi r}\, p(\xi)\, \frac{m_b^2 - 1}{m_b^2 \cdot E_b} \cdot d\xi \cdot d\zeta \qquad \text{- - - (22)}$$

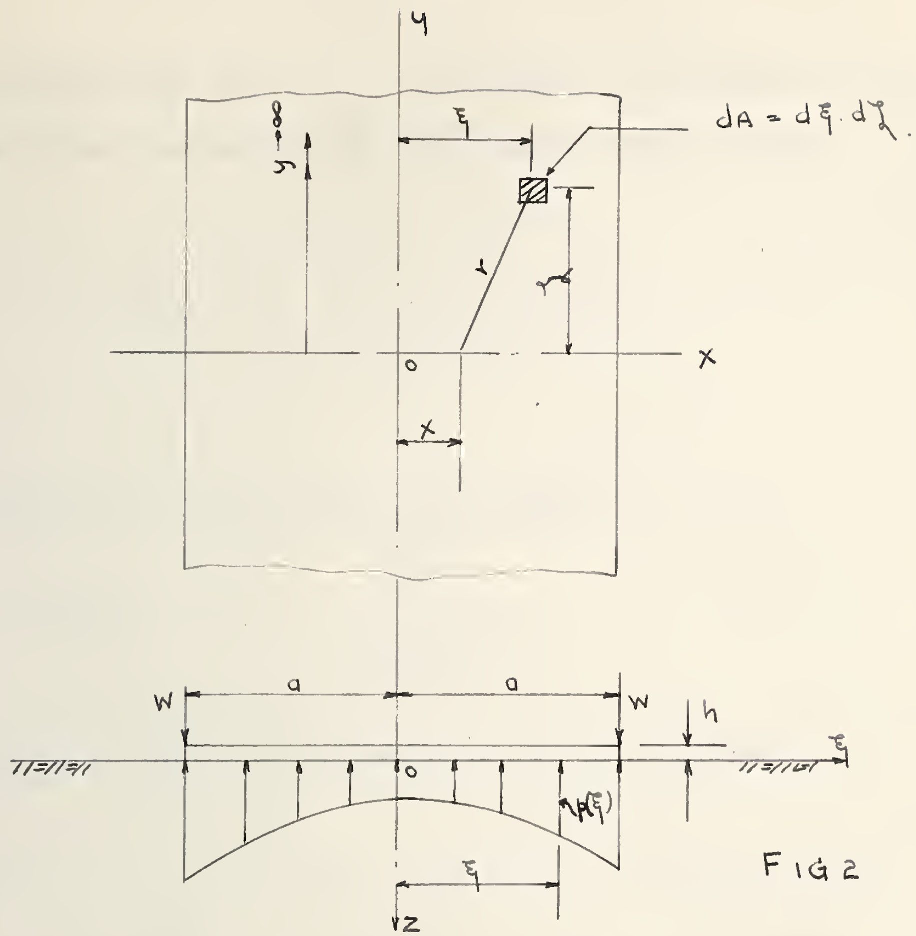

FIG 2

where the new variables as shown in figure 2 are represented by ξ and ζ

In equation (22) let

$$N = \frac{m_b^2 E_b}{m_b^2 - 1}$$

and replace r by $\sqrt{(\xi - x)^2 + \zeta^2}$ Considering first the variable ζ we have,

$$dw = \frac{2}{\pi N} p(\xi)\, d\xi \cdot \int_0^y \frac{d\zeta}{\sqrt{(\xi - x)^2 + \zeta^2}}$$

$$dw = \frac{2}{\pi N} p(\xi)\, d\xi \left[\log\left(y + \sqrt{(\xi - x)^2 + y^2} \right) - \log(\xi - x) \right]$$

Since dw becomes large when $y \to \infty$ we consider the difference in the deflection between any point x and the origin, thus we obtain

$$dw_0 - dw = \frac{2}{\pi N} p(\xi)\, d\xi \cdot \left[\log \frac{y + \sqrt{\xi^2 + y^2}}{y + \sqrt{(\xi - x)^2 + y^2}} + \log|\xi - x| - \log|\xi| \right]$$

In the limit as $y \to \infty$ the first term reduces to

$$\lim_{y \to \infty} \log \frac{1 + \sqrt{\left(\frac{\xi}{y}\right)^2 + 1}}{1 + \sqrt{\left(\frac{\xi - x}{y}\right)^2 + 1}} = \log 1 = 0$$

Integrating with respect to the variable ξ we then have

$$w_0 - w = \frac{2}{\pi N} \int_{-a}^{a} p(\xi) \left[\log|\xi - x| - \log|\xi| \right] d\xi \quad --- (23)$$

$$= \frac{2}{\pi N} \left[\int_{-a}^{x} p(\xi) \log(x - \xi)\, d\xi + \int_{x}^{a} p(\xi) \log(\xi - x)\, d\xi - 2\int_{0}^{a} p(\xi) \log|\xi|\, d\xi \right] -- (24)$$

To evaluate the first two integrals in (24) we define

$$F(\xi) = \int_{x}^{\xi} p(\xi)\, d\xi = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n+1)} \frac{1}{a^{2n}} \left(\xi^{2n+1} - x^{2n+1} \right)$$

and $F(x) = 0$

Integrating I_1 by parts we have that

$$I_1 = \int_{-a}^{x} p(\xi)\log(x-\xi)\,d\xi$$

$$= -\Big[F(\xi)\log(x-\xi)\Big]_{x}^{-a} + \int_{x}^{-a} \frac{F(\xi)\,d\xi}{\xi - x}.$$

$$I_1 = -F(-a)\log(x+a) - \int_{-a}^{x} \frac{F(\xi)\,d\xi}{(\xi - x)}$$

Also for I_2 we have

$$I_2 = \int_{x}^{a} p(\xi)\log(\xi - x)\,d\xi.$$

$$= \Big[F(\xi)\log(\xi - x)\Big]_{x}^{a} - \int_{x}^{a} \frac{F(\xi)\,d\xi}{\xi - x}.$$

$$I_2 = F(a)\log(a-x) - \int_{x}^{a} \frac{F(\xi)\,d\xi}{\xi - x}.$$

Adding I_1 and I_2 we find that

$$I_1 + I_2 = F(a)\log(a-x) - F(-a)\log(a+x) - \int_{-x}^{a} \frac{F(\xi)}{\xi - x}\,d\xi$$

$$I_1 + I_2 = \sum_{n=0}^{\infty} \frac{B_{2n}}{2n+1}\left[\frac{a^{2n+1} - x^{2n+1}}{a^{2n}}\log(a-x) + \frac{a^{2n+1} + x^{2n+1}}{a^{2n}}\log(a+x)\right] - \frac{1}{a^{2n}}\int_{-a}^{a} \frac{\xi^{2n+1} - x^{2n+1}}{\xi - x}\,d\xi \quad \text{--}(25)$$

Expanding the last term in (25) we see that

$$\frac{\xi^{2n+1} - x^{2n+1}}{\xi - x} = \sum_{r=0}^{2n} \xi^{2n-r} x^{r}$$

which upon substitution in (25) gives

$$I_1 + I_2 = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n+1)\,a^{2n}} \Big[\left(a^{2n+1} - x^{2n+1} \right) \log (a-x)$$

$$+ \left(a^{2n+1} + x^{2n+1} \right) \log (a+x) - \sum_{r=0}^{2n} x^r \left(\frac{a^{2n-r+1} - (-a)^{2n-r+1}}{2n-r+1} \right) \Big]$$

From the last integral I_3 in (24) we obtain

$$I_3 = 2 \int_0^a p(\xi) \log |\xi| \, d\xi = 2 \sum_{n=0}^{\infty} \frac{B_{2n}}{2n+1} \, a \left(\log a - \frac{1}{2n+1} \right)$$

Adding and cancelling we have

$$w_0 - w = \frac{2}{\pi N} \sum_{n=0}^{\infty} \frac{B_{2n}}{2n+1} \, a \Big[\log \left(1 - \frac{x}{a}\right) + \log \left(1 + \frac{x}{a}\right) - \left(\frac{x}{a}\right)^{2n+1} \Big(\log \left(1 - \frac{x}{a}\right)$$

$$- \log \left(1 + \frac{x}{a}\right) \Big) - 2 \sum_{r=1}^{n} \frac{1}{2n-2r+1} \left(\frac{x}{a}\right)^{2r} \Big]$$

Expressing the logarithmic terms by their series expansion we write

$$w_0 - w = \frac{2}{\pi N} \sum_{n=0}^{\infty} \frac{B_{2n}}{2n+1} \, a. \Big[- \sum_{r=1}^{\infty} \frac{1}{r} \left(\frac{x}{a}\right)^r$$

$$- \sum_{r=1}^{\infty} \frac{(-1)^r}{r} \left(\frac{x}{a}\right)^r - \left(\frac{x}{a}\right)^{2n+1} \sum_{r=1}^{\infty} \frac{(-1)^r}{r} \left(\frac{x}{a}\right)^r$$

$$+ \left(\frac{x}{a}\right)^{2n+1} \sum_{r=1}^{\infty} \frac{1}{r} \left(\frac{x}{a}\right)^r - \sum_{r=1}^{n} \frac{1}{2n-2r+1} \left(\frac{x}{a}\right)^{2r} \Big]$$

Adding even and odd powers the expression for the deflection of the foundation reduces to

$$w_0 - w = \frac{4a}{\pi N}\left[\sum_{n=0}^{\infty} \frac{B_{2n}}{2n+1}\left[\left(\frac{x}{a}\right)^{2n} \sum_{r=1}^{\infty} \frac{1}{2r-1}\left(\frac{x}{a}\right)^{2r} - \sum_{r=1}^{\infty} \frac{1}{2r}\left(\frac{x}{a}\right)^{2r} - \sum_{r=1}^{n} \frac{1}{2n-2r+1}\left(\frac{x}{a}\right)^{2r}\right]\right]$$

or

$$w_0 - w = \frac{4a}{\pi N}\left[\sum_{r=1}^{\infty}\sum_{n=0}^{\infty} \frac{B_{2n}}{(2r-1)(2n+1)}\left(\frac{x}{a}\right)^{2n+2r} - \sum_{r=1}^{\infty}\sum_{n=0}^{\infty} \frac{B_{2n}}{2r(2n+1)}\left(\frac{x}{a}\right)^{2r} - \sum_{n=0}^{\infty}\sum_{r=1}^{n} \frac{B_{2n}}{(2n-2r+1)(2n+1)}\left(\frac{x}{a}\right)^{2r}\right]$$

Lettering $r' = r-1$ and then rewriting r instead of r' we have

$$w_0 - w = \frac{4a}{\pi N}\left[\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{B_{2n}}{(2r+1)(2n+1)}\left(\frac{x}{a}\right)^{2n+2r+2} - \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{B_{2n}}{(2r+2)(2n+1)}\left(\frac{x}{a}\right)^{2r+2} - \sum_{n=0}^{\infty}\sum_{r=1}^{n} \frac{B_{2n}}{(2n-2r+1)(2n+1)}\left(\frac{x}{a}\right)^{2r}\right] \quad \text{---(26)}$$

The first summation expression in (26) can be rewritten in terms of one variable index by summing along diagonals first from $n=0$ to k and then from $k=0$ to ∞ as shown in figure 3.

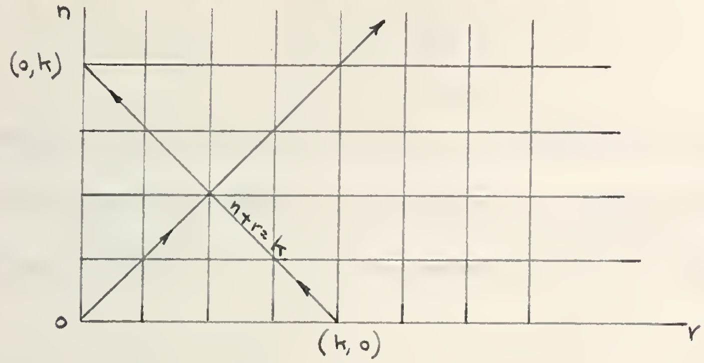

FIG 3

The final expression for the deflection of the upper surface of the foundation due to the load $p(x)$ is

$$w_0 - w = \frac{4q}{\pi N}\left[\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{B_{2n}}{(2k-2n+1)(2n+1)}\left(\frac{x}{a}\right)^{2k+2}\right.$$

$$\left.-\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{B_{2n}}{(2r+2)(2n+1)}\left(\frac{x}{a}\right)^{2r+2}-\sum_{n=0}^{\infty}\sum_{r=1}^{n}\frac{B_{2n}}{(2n-2r+1)(2n+1)}\left(\frac{x}{a}\right)^{2r}\right] \quad ---(27)$$

Expression (26) for the deflection of the foundation agrees with Borowicka's equation (9)[7]. The coefficient for $\left(\frac{x}{a}\right)^{2m}$ can be written from (27) as

$$\frac{4q}{N}\left[\sum_{m=0}^{m-1}\frac{B_{2n}}{(2m-2n-1)(2n+1)}-\sum_{n=0}^{\infty}\frac{B_{2n}}{(2n+1)(2m)}\right.$$

$$\left.-\sum_{n=m}^{\infty}\frac{B_{2n}}{(2n-2m+1)(2n+1)}\right]$$

The condition for equilibrium for case 1 is

$$W = \sum_{n=0}^{\infty} \int_0^a B_{2n} \left(\frac{x}{a}\right)^{2n} dx$$

or

$$\frac{W}{a} = \sum_{n=0}^{\infty} \frac{B_{2n}}{2n+1} \qquad \text{- - - - (28)}$$

For the deflection of the slab and the foundation to be compatible equations (21 and (27) must be identical. This condition is fulfilled when the coefficients of all like powers of $\frac{x}{a}$ are equal. Writing

$$K = \frac{2D}{Na^3} = \frac{1}{6} \; \frac{m_b^2 - 1}{m_s^2 - 1} \cdot \frac{m_s^2}{m_b^2} \cdot \frac{E_s}{E_b} \cdot \left(\frac{h}{a}\right)^3$$

we have from the coefficients of $\left(\frac{x}{a}\right)^2$

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n+1)(1-2n)} + \frac{\pi}{4K} \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n+2)(2n+1)} = \frac{W}{2a}. \qquad \text{- - - (29)}$$

and for the general term $(x/a)^{2m}$ where $m \geq 2.$

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{(2m-2n-1)(2n+1)} - \frac{\pi}{2K} B_{2m-4} \frac{(2m-4)!}{2m!} = \frac{W}{2ma} \qquad \text{- - (30)}$$

By splitting the terms in equations (29) and (30) into partial fractions and substituting

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{2n+1} \quad \text{for} \quad \frac{W}{a} \quad \text{and}$$

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{2n+1} \cdot \frac{1}{2m} \quad \text{for} \quad \frac{W}{2ma}.$$

we have the following homogenous equations along with (28) for the solution of the B's.

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{1-2n} + \frac{\pi}{4K} \sum_{n=0}^{\infty} \frac{B_{2n}}{(n+1)(2n+1)} = 0 \qquad \text{---} (31a)$$

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{2m-2n-1} - \frac{\pi}{2K} B_{2m-4} \frac{(2m-4)!}{(2m-1)!} = 0 \qquad \text{---} (31b) \qquad m \geq 2$$

Equations (31) can be solved in terms of B_0 and the final expression for the coefficients can be obtained in terms of the load and the half width of the slab from (28). The value of the resulting coefficients substituted in the expression

$$p(x) = \sum_{n=0}^{\infty} B_{2n} \left(\frac{x}{a}\right)^{2n}.$$

gives the required formula for the determination of the contact pressure of a concentrated line load at the edge of the slab when the B_{2n}'s are known.

<u>Case 2</u>

Overturning Moments at the Edge of the Slab

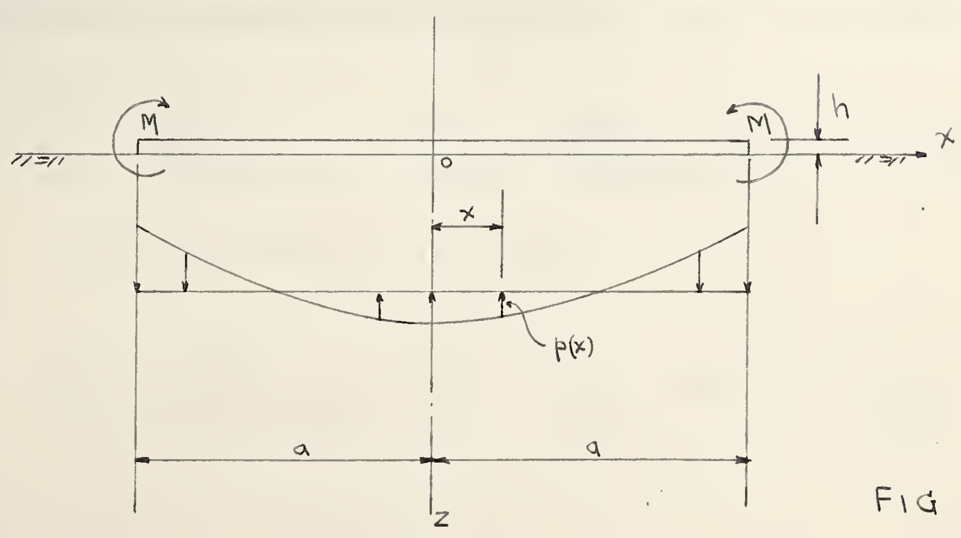

FIG 4

For the case of the overturning moments shown in figure 4 the boundry conditions for the upper surface of the foundation are

$$x > a \qquad \sigma_z = 0$$

$$x < a \qquad \sigma_z = p(x).$$

For the slab we have

$$x = a \qquad \frac{d^2 w}{dx^2} = -\frac{M}{D}$$

$$x = 0 \qquad w = w_0 \quad ; \quad \frac{dw}{dx} = 0 \quad ; \quad \frac{d^3 w}{dx^3} = 0.$$

The slab equation for case 2 is

$$\frac{d^4 w}{dx^4} = -\frac{p(x)}{D}$$

Proceeding in a manner similar to that for the concentrated loads it results that the deflection of the slab can be expressed as

$$w - w_0 = -\left(\frac{a^2 M}{2D} - \frac{a^4}{2D} \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n+2)(2n+1)} \right) \left(\frac{x}{a}\right)^2 - \frac{a^4}{D} \sum_{n=0}^{\infty} B_{2n} \frac{2n!}{(2n+4)!} \left(\frac{x}{a}\right)^{2n+4} \quad \text{---} (32)$$

The deflection of the foundation is given by equation (27). Comparing coefficients of like powers of x/a we obtain from (27) and (32) for $(x/a)^2$

$$\left.\begin{aligned} &\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n+1)(1-2n)} + \frac{\pi}{4K} \frac{B_{2n}}{(2n+2)(2n+1)} = \frac{\pi M}{4 K a^2} \\ &\text{and in general for } (x/a)^{2m} \\ &\sum_{n=0}^{\infty} \frac{B_{2n}}{(2m-2n-1)(2n+1)} - \frac{\pi}{2K} B_{2m-4} \frac{(2m-4)!}{2m!} = 0 \end{aligned}\right\} \quad \text{--} (32)$$

The condition of equilibrium requires that

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{2n+1} = 0 \qquad \text{---(33)}$$

Breaking the terms in equations (32) into partial portions and combining with (33) the B's in terms of B_0 can be determined from (33) and the following homogenous equations

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{2m-2n-1} - \frac{\pi}{2K} B_{2m-4} \frac{(2m-4)!}{(2m-1)!} = 0 \qquad \text{---(34)}$$

where $m \geq 2$ The final value of the coefficients in terms of M, a and K can be determined by substitution in (35)

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{1-2n} - \frac{\pi}{4K} \sum_{n=0}^{\infty} \frac{B_{2n}}{n+1} = \frac{\pi}{2K} \frac{M}{a^2} \qquad \text{---(35)}$$

The resulting expression for the coefficients placedin the equation

$$p(x) = \sum_{n=0}^{\infty} B_{2n} \left(\frac{x}{a}\right)^{2n}$$

will give the required formula for the determination of the contact pressure.

Case 3

The Uniform Load

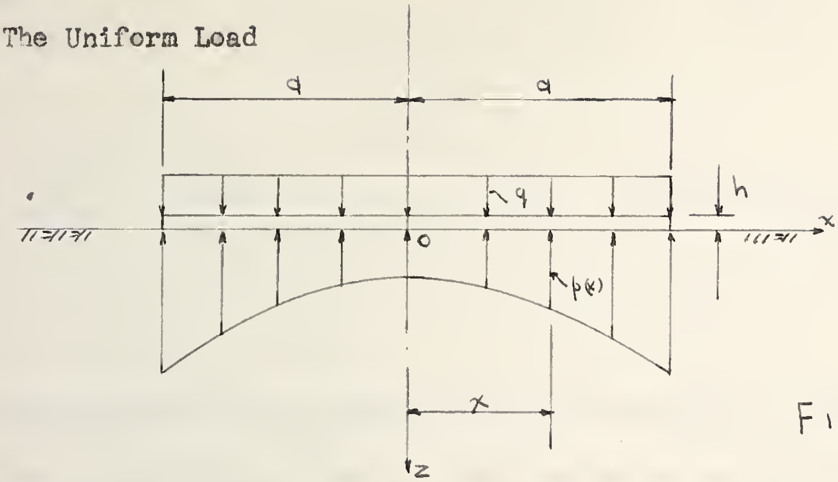

FIG 5.

This case has been treated by Borowicka[7] and the resulting equations for the solution of the B's are

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{2n+1} = q \qquad \text{--- (36)}$$

$$\left.\begin{aligned}
&\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n+1)(1-2n)} + \frac{\pi}{4K} \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n+2)(2n+1)} = \frac{q}{2} + \frac{\pi}{8K} q \\
&\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n+1)(3-2n)} - \frac{\pi}{48K} B_0 = \frac{q}{4} - \frac{\pi}{48K} q \\
&\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
&\sum_{n=0}^{\infty} \frac{B_{2n}}{2m-2n-1} - \frac{\pi}{2K} B_{2m-4} \frac{(2m-4)!}{2m!} = \frac{q}{2m}
\end{aligned}\right\} \qquad \text{--- (37)}$$

where $m \geq 3$ By breaking the summation expressions in (37) into partial fractions and combining with the condition of equilibrium (36) we have the following homogenous equations (38) and (39) for the determination of the B's in terms of B_0:

$$\sum_{n=0}^{\infty} B_{2n} \frac{n}{(1-2n)(3-2n)} - \frac{\pi}{16K} \sum_{n=1}^{\infty} \frac{B_{2n}}{n+1} = 0 \quad \text{--- (38)}$$

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{2m-2n-1} - \frac{\pi}{2K} B_{2m-4} \frac{(2m-4)!}{(2m-1)!} = 0 \quad \text{--- (39)}$$

The B's are finally evaluated in terms of the load q from the equation of equilibrium (36). Substitution of the resulting coefficients in the equation

$$p(x) = \sum_{n=0}^{\infty} B_{2n} \left(\frac{x}{a}\right)^{2n}$$

will establish the expression necessary to determine the contact pressure distribution.

The labour require to determine the coefficients for cases 1 and 2 can be reduced by realizing that the group of equations given by

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{(2m-2n-1)} - \frac{\pi}{2K} B_{2m-4} \frac{(2m-4)!}{(2m-1)!} = 0 \qquad m \geq 2$$

is common to both solutions. A group of these equations for any given K value could be solved in terms of B_0 and B_2, then by substitution in the two remaining equations, a solution of the unknown coefficients in terms of the load results. The two remaining equations for case 1 are (31)(a) and (28), and for case 2 (35) and (33).

Discussion of Borowicka's Solution

In attempts to determine the solution for the contact pressure of symmetrically placed concentrated or discontinuous uniform loads it was found that sufficient equations could not be obtained to compare all like powers of x/a. It was found that the equation for the deflection of the slab contained even and odd powers of x whereas that for the foundation had only terms in even powers of x and a term in log x/a

To indicate the reason for this discrepancy, the integral involving the deflection of the foundation is first considered. The deflection $\omega(x)$ is expressed as

$$\omega(x) = \int_0^1 p(\xi)\left[\log|\xi - x| + \log(\xi + x) - 2\log|\xi|\right] d\xi.$$

expanding we have

$$\omega(x) = \int_0^x p(\xi)\log(x-\xi)\,d\xi + \int_x^1 p(\xi)\log(\xi - x)\,d\xi$$

$$+ \int_0^1 p(\xi)\log(\xi + x)\,d\xi - 2\int_0^1 p(\xi)\log\xi\,d\xi.$$

Defining the integral

$$\int_x^\xi p(\xi)\,d\xi = F(\xi) - F(x)$$

we have integrating I_1 by parts

$$I_1 = \int_0^1 p(\xi)\log|\xi - x|\,d\xi.$$

$$= \left(F(\xi) - F(x)\right)\log(x-\xi)\Big|_0^x - \int_0^x \frac{F(\xi) - F(x)}{\xi - x}\,d\xi$$

$$+ \left(F(\xi) - F(x)\right)\log(\xi - x)\Big|_x^1 - \int_x^1 \frac{F(\xi) - F(x)}{\xi - x}\,d\xi.$$

$$I_1 = -\left(F(0) - F(x) \right) \log x . + \left(F(1) - F(x) \right) \log (1-x)$$

$$- \int_0^1 \frac{F(\xi) - F(x)}{\xi - x} \, d\xi \quad \text{---(40)}$$

For the seond integral we have, integrating by parts

$$I_2 = \int_0^1 p(\xi) \log (\xi + x) \, d\xi .$$

$$= F(\xi) \log (\xi + x) \Big|_0^1 - \int_0^1 \frac{F(\xi) \, d\xi}{\xi + x} .$$

$$I_2 = F(1) \log (1+x) - F(0) \log x - \int_0^1 \frac{F(\xi) \, d\xi}{\xi + x} \quad \text{---(41)}$$

Expanding $F(\xi)$ in the last term of (41) in a Taylor series about (-x), we have

$$F(\xi) = F(-x) + F(-x)' (\xi + x) + F''(-x) \frac{(\xi + x)^2}{2!} + \text{----}$$

Substituting the expanded func~~t~~tion back in (41) we can write that

$$I_2 = F(1) \log (1+x) - F(0) \log x - \int_0^1 \left(\frac{F(-x)}{\xi + x} + F'(-x) \right.$$

$$\left. + F''(-x) \frac{(\xi + x)}{2!} + \text{---} . \right) d\xi .$$

which upon integration gives

$$I_2 = F(1) \log (1+x) - F(0) \log x . - \Big[F(-x) \log (\xi + x) \Big]_0^1 + \text{------}$$

$$I_2 = F(1) \log (1+x) - F(0) \log x - F(-x) \log (1+x) + F(-x) \log x + \text{----}$$

Collecting terms in I_1 and I_2 we have

$$\omega(x) = \log x \left[F(x) + F(-x) - 2F(0) \right]$$

$$+ \log(1-x) \left[F(1) - F(x) \right]$$

$$+ \log(1+x) \left[F(1) - F(-x) \right]$$

$$- \int_0^1 \frac{F(\xi) - F(x)}{\xi - x} \, d\xi \;-\; 2\int_0^1 p(\xi) \log \xi \, d\xi$$

$$+ - - - - -$$

When dealing with loads that are continuous over the full width of the slab and are symmetrically placed, the contact pressure $p(x)$ can be defined in even powers of x/a for $x \gtrless 0$ that is

$$p(x) = \sum_{n=0}^{\infty} B_{2n} \left(\frac{x}{a}\right)^{2n} \qquad x > 0.$$

$$p(x) = \sum_{n=0}^{\infty} B_{2n} \left(\frac{x}{a}\right)^{2n} \qquad x < 0$$

Then $F(\xi)$ which was defined as

$$F(\xi) = \int_x^{\xi} p(\xi) \, d\xi \qquad x \gtrless 0$$

can be written in series form as

$$F(x) = C_0 + C_2 x^2 + C_4 x^4 + - - - \qquad x > 0$$

$$F(x) = C_0 + C_2 x^2 + C_4 x^4 + - - - \qquad x < 0$$

In the limit as $x \to 0$ we have that the

$$\lim_{x \to 0} F(x) = C_0 \qquad \text{for} \quad x \gtrsim 0$$

Now considering equation (42) for w(x) we see that the log x term disappears and that precisely the same terms which appeared in Borowicka's solution remain. This argument then establishes the validity of the use of a Maclaurin series with even powers to represent the contact pressure distribution for continuous symmetrical loadings.

Let us now consider the case of the symmetrical but discontinuous loading. For example the concentrated load at the centre of the slab. The contact pressure is now defined by both even and odd powers of x as

$$p(x) = \sum_{n=0}^{\infty} a_n x^n \qquad x > 0$$

$$p(-x) = \sum_{n=0}^{\infty} a_n (-x)^n \qquad x < 0$$

We defined

$$F(\xi) = \int_x^{\xi} p(\xi)\, d\xi \qquad x > 0$$

$$F(-\xi) = \bar{F}(\xi) = \int_x^{\xi} p(-\xi)\, d\xi \qquad x < 0$$

or in series form we can write

$$F(x) = C_0 + C_1 x + C_2 x^2 + \text{- - - -} \qquad x > 0$$

$$F(-x) = -C_0 - C_1 x + C_2 x^2 - \text{- - - -} \qquad x < 0$$

In the limit as $x \to 0$ we have that

$$\lim_{x \to 0} F(x) = C_0 \qquad x > 0$$

$$\lim_{x \to 0} F(-x) = -C_0 \qquad x < 0$$

An example of a function that has such a discontinuity is the shear under a concentrated load. The log x term in equation (42) no longer cancels. This implies that there is some form of singularity at the origin. The assumption then that the contact pressure p(x) can be expressed as a power series containing even and odd positive powers of x is not valid. Therefore for discontinuous symmetrical loads a different series expansion must be considered or a different method of analysis investigated.

It should be noted that substitution of $x' = 1 - x$ or $x' = 1 + x$ in (42) would not remove the singularity that occurs.

Inverse Problem

A consideration of what is termed the inverse problem leads to a determination of what superimposed load should be placed on the slab to produce a prescribed deflection curve or a prescribed contact pressure distribution. Or fundamentally what physical interpretation can be attached to an assumed expression for the contact pressure or the deflection curve.

Let us first consider the case where the contact pressure is assumed and is of the form

$$p(x) = \mathcal{L}\left(\frac{x}{a}\right)^s \qquad \text{- - - - (43)}$$

where s is a positive integer. The deflection of the foundation can be computed from Bouessesq's equation (12), and proceeding as under case 1 we have that

$$\omega_0 - \omega = \frac{2}{\pi N}\left[\int_{-a}^{a} p(\xi)\log|\xi - x|\,d\xi - \int_{-a}^{a} p(\xi)\log|\xi|\,d\xi\right]$$

and upon substitution of (43) and completion of the integration the expression for the deflection reduces to

$$w_0 - w = \frac{4\pi\alpha}{N}\,\frac{a}{2s+1}\left[\sum_{r=1}^{\infty}\frac{1}{2r-1}\left(\frac{x}{a}\right)^{2s+2r} - \sum_{r=1}^{\infty}\frac{1}{2r}\left(\frac{x}{a}\right)^{2r} - \sum_{k=1}^{s}\frac{1}{2s-2k+1}\left(\frac{x}{a}\right)^{2k}\right]_{(s\ \text{even})}$$

$$+\frac{4\pi\alpha}{N}\,\frac{a}{2s}\left[\sum_{r=1}^{\infty}\frac{1}{2r}\left(\frac{x}{a}\right)^{2s+2r} - \sum_{r=1}^{\infty}\frac{1}{2r}\left(\frac{x}{a}\right)^{2r} - \sum_{k=1}^{s}\frac{1}{2s-2k}\left(\frac{x}{a}\right)^{2k} + \left(\frac{x}{a}\right)^{2s}\log\frac{x}{a}\right]_{(s,\ \text{odd})} \quad ---(44)$$

The terms in the first square bracket of expression (44) apply only when s is an even integer, and those in the second bracket apply only when s is odd. The equation for the determination of the superimposed load q(x) is obtained by first differentiating (44) with respect to x and then substituting in the slab equation

$$D\,\frac{d^4 w}{d x^4} = q(x) - p(x)$$

As a first example consider the case when $p(x)$ is a constant α that is $s = 0$ We have then from (44)

$$w_0 - w = \frac{2\pi}{N}\,\alpha\, a \sum_{r=1}^{\infty}\frac{1}{2r}\left(\frac{x}{a}\right)^{2r} \quad ----(45)$$

Differentiating (45) and summing the series we have

$$\frac{d^4 w}{d x^4} = -\frac{8\pi}{N a^3}\,\alpha\,\frac{1-3\left(\frac{x}{a}\right)^2}{\left(1-\left(\frac{x}{a}\right)^2\right)^3} \quad \text{-----} (46)$$

Substitution of (46) in the slab equations gives the expression for the superimposed load on the slab as follows

$$q(x) = \alpha\left[1 - 4\pi K\,\frac{1-3\left(\frac{x}{a}\right)^2}{\left(1-\left(\frac{x}{a}\right)^2\right)^3}\right] \quad \text{---} (47)$$

or

$$q(x) = \alpha\left[1 - C_0 f_0\left(\frac{x}{a}\right)\right]$$

where $C_0 = 4\pi K$. and $f_0\left(\frac{x}{a}\right) = \dfrac{1-3\left(\frac{x}{a}\right)^2}{\left(1-\left(\frac{x}{a}\right)^2\right)^3}$

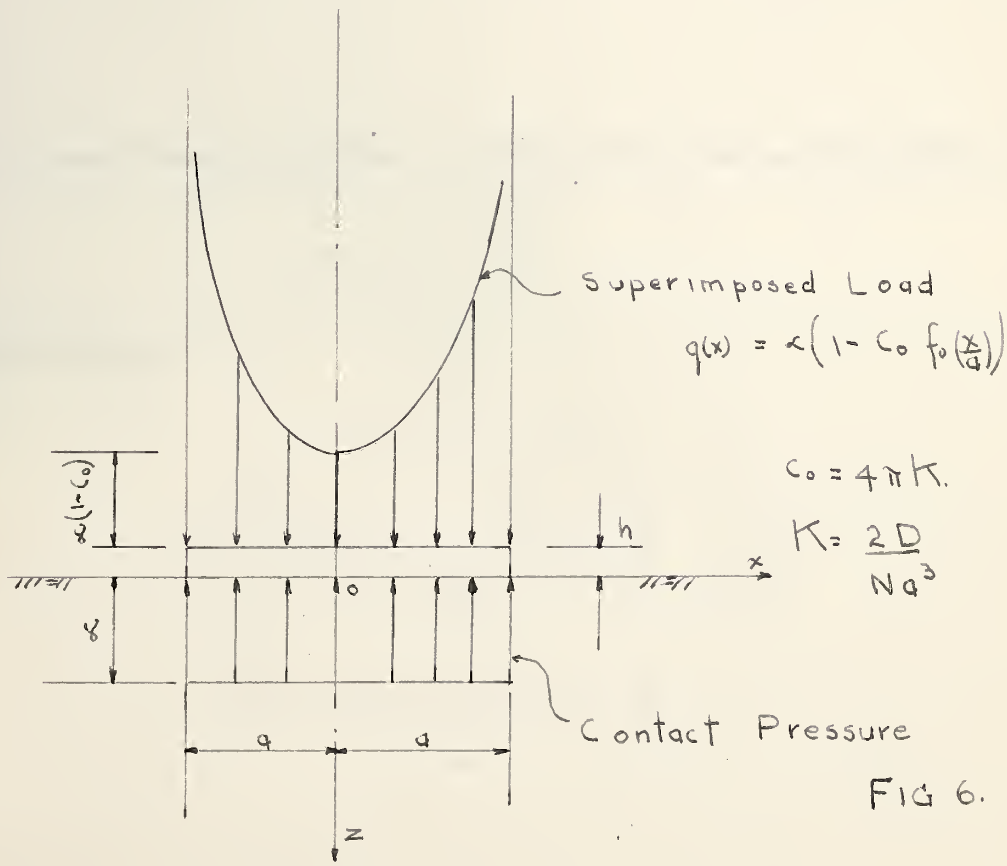

FIG 6.

The distribution of the superimposed load when the foundation reaction is assumed constant is shown in figure 6.

As a second example consider the problem when the contact pressure is

$$p(x) = \beta \left| \frac{x}{a} \right| .$$

as shown in figure 7.

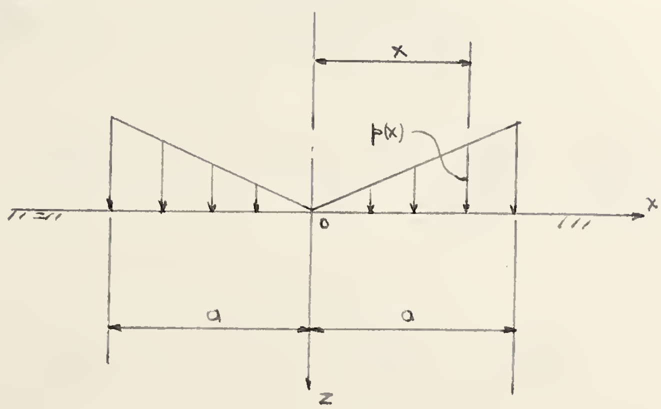

FIG 7.

From equation (44) we have for $s = 1$

$$\omega_0 - \omega = \frac{2\pi a}{N}\beta\left[\left(\frac{x}{a}\right)^2 \log\frac{x}{a} - \sum_{r=1}^{\infty}\frac{1}{2r}\left(\frac{x}{a}\right)^{2r} + \sum_{r=1}^{\infty}\frac{1}{2r}\left(\frac{x}{a}\right)^{2r+2}\right] \quad \text{---} (48)$$

By summing the last two series expressions, equation (48) can be written in simpler form as

$$\omega_0 - \omega = \frac{2\pi a}{N}\beta\left[\mu^2 \log\mu + \frac{1}{2}\log(1-\mu^2) - \frac{\mu^2}{2}\log(1-\mu^2)\right] \quad \text{----} (49)$$

where $\mu = x/a$ Taking the fourth derivate of (49) with respect to μ we have

$$a^4\frac{d^4\omega}{dx^4} = \frac{4\pi a}{N}\beta\left[\frac{1 - 6\left(\frac{x}{a}\right)^2 - 3\left(\frac{x}{a}\right)^4}{\left(\frac{x}{a}\right)^2\left(1-\left(\frac{x}{a}\right)^2\right)^3}\right]$$

Substituting in the slab equation we have for the required superimposed load

$$q(x) = \beta\left[2\pi K\left[\frac{1 - 6\left(\frac{x}{a}\right)^2 - 3\left(\frac{x}{a}\right)^4}{\left(\frac{x}{a}\right)^2\left(1-\left(\frac{x}{a}\right)^2\right)^3}\right] + \left|\frac{x}{a}\right|\right]$$

$$q(x) = \beta\left[c_1 f_1\left(\frac{x}{a}\right) + \left|\frac{x}{a}\right|\right] \quad \text{---} (50)$$

where $c_1 = 2\pi K$.

The superimposed load for an assumed linear variation of the contact pressure approaches $+\infty$ when $\frac{x}{a} \to 0$ and $-\infty$ when $\frac{x}{a} \to 1$

A plot of equation (50) is shown in figure 8. This solution is of practical interest when combined with others to give an approximation to more practical problems such as the concentrated load.

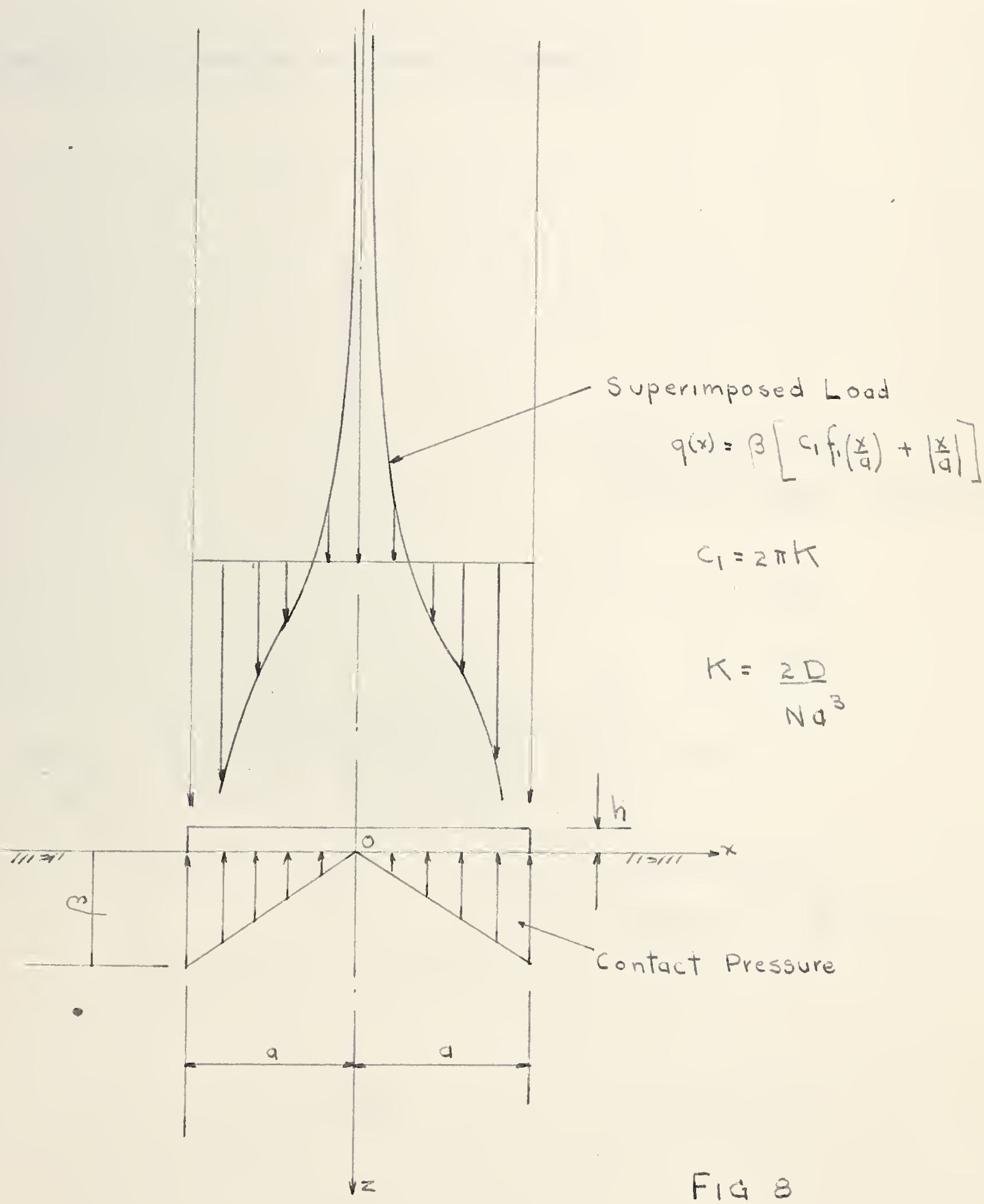

FIG 8

By applying the principle of superposition to the two foregoing examples we obtain such an approximation for the concentrated load. The load on the slab is given as

$$q(x) = \alpha \left[c_1 f_1 \left(\frac{x}{a}\right) + c_0 f_0 \left(\frac{x}{a}\right) + \left|\frac{x}{a}\right| - 1 \right] \quad \text{---} (51)$$

and the graph of the function is shown in figure 9.

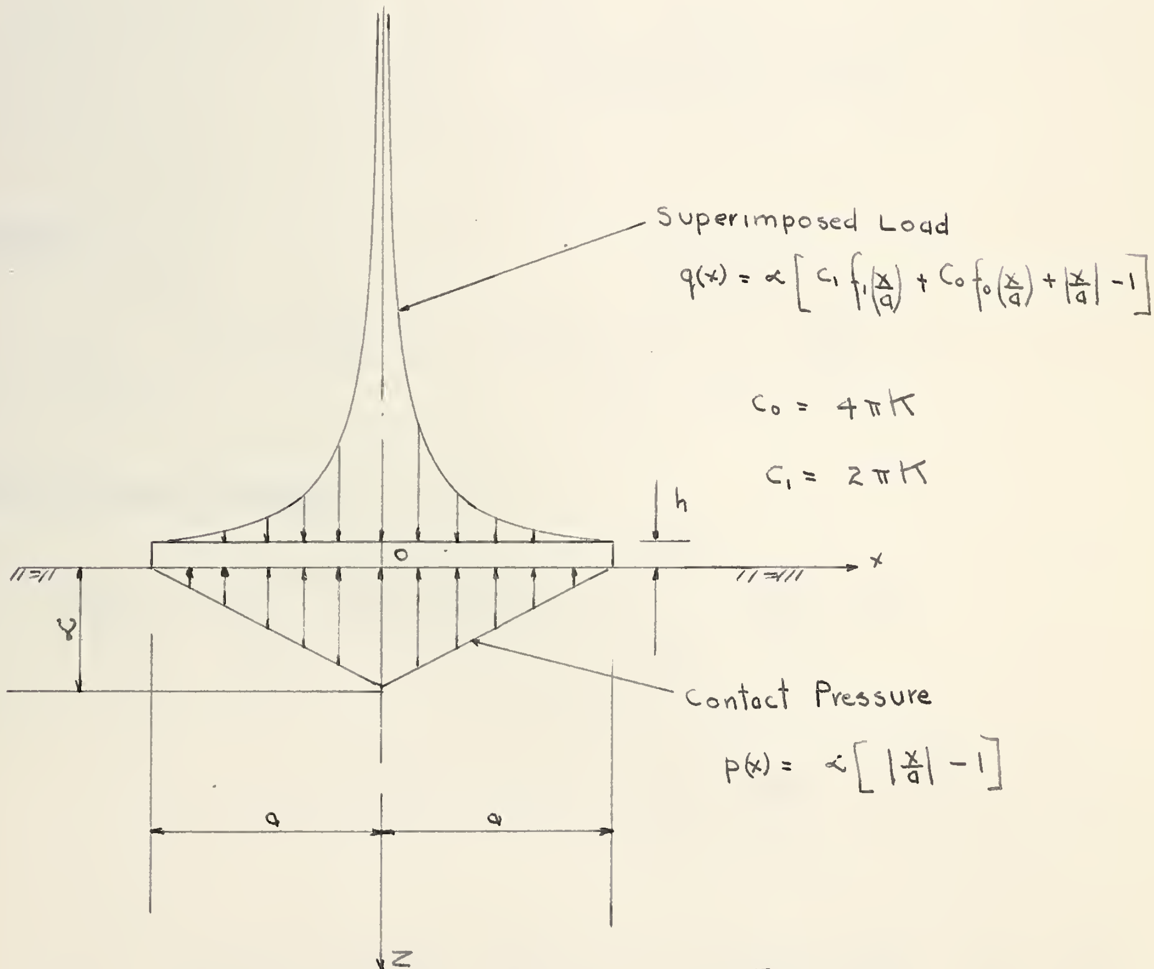

FIG 9.

Let us now consider problems for which the deflection of the slab or foundation is some prescribed function. Considering the deflection of the foundation

$$\omega_0 - \omega(x) = \frac{2}{\pi N}\int_{-1}^{1} p(\xi)\Big[\log|\xi - x| - \log|\xi|\Big]\,d\xi.$$

which arises from Boussinesq's equation (12), we have on differentiating with respect to x that

$$-\frac{d\omega}{dx} = \frac{2}{\pi N}\frac{d}{dx}\Big[\int_{-1}^{x-\varepsilon} p(\xi)\log(x-\xi)\,d\xi.$$

$$+\int_{x+\varepsilon}^{1} p(\xi)\log(\xi - x)\,d\xi\Big]$$

Recalling that

$$\frac{d}{dx}\int_{a}^{x_1} F(x,\xi)\,d\xi = F(x,x_1) + \int_{a}^{x_1}\frac{\partial F}{\partial x}\,d\xi.$$

we have for the above integral

$$-\frac{d\omega}{dx} = \frac{2}{\pi N}\Big[p(x-\varepsilon)\log\varepsilon + \int_{-1}^{x-\varepsilon}\frac{p(\xi)}{x-\xi}\,d\xi.$$

$$-p(x+\varepsilon)\log\varepsilon + \int_{x+\varepsilon}^{1}\frac{p(\xi)}{x-\xi}\,d\xi.\Big]$$

$$= \frac{2}{\pi N}\Big[\frac{p(x-\varepsilon) - p(x+\varepsilon)}{2\varepsilon}\,.\,2\varepsilon\log\varepsilon + \oint_{-1}^{1}\frac{p(\xi)}{x-\xi}\,d\xi\Big]$$

$$-\frac{dw}{dx} = \frac{2}{\pi N}\left[\, p'(x)\, 2\varepsilon \log \varepsilon + \oint_{-1}^{1} \frac{p(\xi)}{x-\xi}\, d\xi \right.$$

as $\varepsilon \to 0$ we have that

$$\frac{1}{c}\frac{dw}{dx} = \frac{1}{\pi}\oint_{-1}^{1} \frac{p(\xi)}{\xi - x}\, d\xi \qquad \text{--- (52)}$$

where the symbol $\oint$ stands for the principal value of the integral, and $c = \frac{2}{N}$

The solution[8] of (52) is obtained by first transforming the variables by letting

$$\xi = \cos\theta \qquad \text{and} \qquad x = \cos\phi$$

We then have for (52)

$$G(\phi) = \frac{1}{\pi}\oint_{\pi}^{0} \frac{p(\cos\theta)\,(-)\sin\theta\, d\theta}{\cos\theta - \cos\phi}$$

$$G(\phi) = \frac{1}{\pi}\int_{0}^{\pi} \frac{P(\theta)\sin\theta\, d\theta}{\cos\theta - \cos\phi} \qquad \text{--- (53)}$$

Expanding $P(\theta)\sin\theta$ in its Fourier expansion, we have

$$P(\theta)\sin\theta = \sum_{m=0}^{\infty} A_m \cos m\theta \qquad \text{--- (54)}$$

Similarly the function $G(\phi)$ is thought to be developed in the interval $(0, \pi)$ in the following form

$$\sin\phi \;\; G(\phi) = \sum_{m=1}^{\infty} B_m \; \sin m\phi \qquad ---(55)$$

Substituting (54) and (55) in (53) we obtain

$$G(\phi) = \sum_{m=1}^{\infty} B_m \frac{\sin m\phi}{\sin\phi} = \sum_{m=0}^{\infty} A_m \frac{1}{\pi} \int_0^{\pi} \frac{\cos m\theta \, d\theta}{\cos\theta - \cos\phi}$$

therefore

$$\frac{1}{\pi} \int_0^{\pi} \frac{\cos m\theta}{\cos\theta - \cos\phi} \, d\theta = \frac{\sin m\phi}{\sin\phi} \qquad ---(56)$$

and

$$\sum_{m=1}^{\infty} B_m = \sum_{m=1}^{\infty} A_m.$$

that is $B_m = A_m$ and A_0 is arbitary. This then permits a determination of the function $A(\theta)$ with unknown coefficients A_m to be made in terms of the known coefficients B_m.

Let

$$H(\phi) = G(\phi) \; \sin\phi$$

and

$$H(\phi) = \sum_{m=1}^{\infty} A_m \sin m\phi$$

Developing as a sine series, we write

$$\int_0^{\pi} H(\phi) \; \sin n\phi \; d\phi = \sum_{m=1}^{\infty} A_m \int_0^{\pi} \sin m\phi \cdot \sin n\phi \; d\phi.$$

The integral on the r.h.s. is zero when $m \neq n$ and is $\frac{\pi}{2}$ when $m = n$ Therefore it follows that the coefficients A_n can be expressed as

$$A_n = \frac{2}{\pi} \int_0^{\pi} G(\phi) \sin\phi \sin n\phi \, d\phi \qquad \text{---(57)}$$

The unknown coefficients A_n thus determined in terms of the known coefficients of the function $G(\phi)$ give a solution of the reaction func~~c~~tion $P(\theta)$

As a first illustration of this method consider the case when the deflection of the foundation is constant that is

$$w = c_0 \qquad x \gtrless 0$$

From the foregoing work we have

$$G(\phi) = \sum_{m=1}^{\infty} A_m \frac{\sin m\phi}{\sin\phi} \qquad \text{---(58)}$$

Since $G(\phi) = 0$ we see that all the coefficients A_{2m} and A_{2m-1}, $m = 1, 2, 3 \cdots$ are zero. Therefore

$$P(\theta) \sin\theta = A_0$$

and

$$p(\xi) = \frac{A_0}{\sqrt{1 - \xi^2}} \qquad \text{---(59)}$$

Equation (59) agrees with that given by Timoshenko reference 6 page 93, for the pressure distribution under a rigid strip which has undergone a constant deflection.

As a second example consider the case when the deflection is assumed parabolic

$$w(x) = c_2 (1 - x^2) \qquad \text{---(60)}$$

We have

$$\frac{dw}{dx} = -2C_2 x \qquad \text{for all } x.$$

and from (58) we write that

$$-2C_2' \cos\phi \sin\phi = \sum_{m=1}^{\infty} A_m \sin m\phi$$

or

$$-2C_2' \sin 2\phi = A_1 \sin\phi + A_2 \sin 2\phi + ---$$

Equating coefficients we find that

$$A_2 = -C_2' .$$

and

$$A_{2m-1} = A_{2m+2} = 0 \qquad m = 1, 2, 3 ---$$

We can then write for the reaction function

$$p(\cos\theta)\sin\theta = -C_2' \cos 2\theta + A_0$$

where A_0 is an arbitary constant. Transforming to the original variable we find

$$p(\xi) = -C_2' \frac{(2\xi^2 - 1)}{\sqrt{1-\xi^2}} + \frac{A_0}{\sqrt{1-\xi^2}}$$

Taking $p(\xi)$ as finite at $\xi = 1$ we have that $A_0 = C_2'$ and the resulting expression for the contact pressure is

$$p(\xi) = 2C_2' \frac{1-\xi^2}{\sqrt{1-\xi^2}}$$

or

$$p(x) = 2C_2' \sqrt{1-x^2} \qquad --- (61)$$

By substituting (60) and (61) in the slab equation we have that

$$\frac{d^4 w}{d x^4} = 0$$

and therefore $$q(x) = p(x) = 2C_2' \sqrt{1-x^2}$$

That is the superimposed load and the contact pressure are equal and opposite and form respectively the upper and lower halves of an ellipse as shown in figure 10.

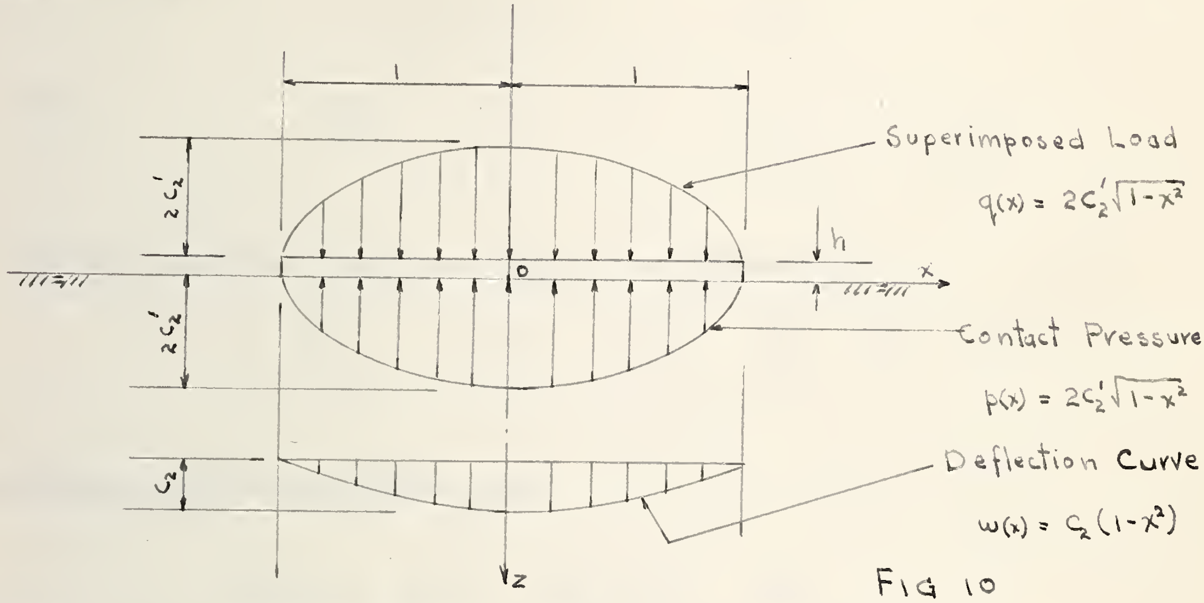

FIG 10

As a final illustration of this method of analysis, consider the problem when the deflection of the foundation is linear and symmetrical about the centreline, that is

$$w(x) = C_1'(1-|x|) \qquad ---(62)$$

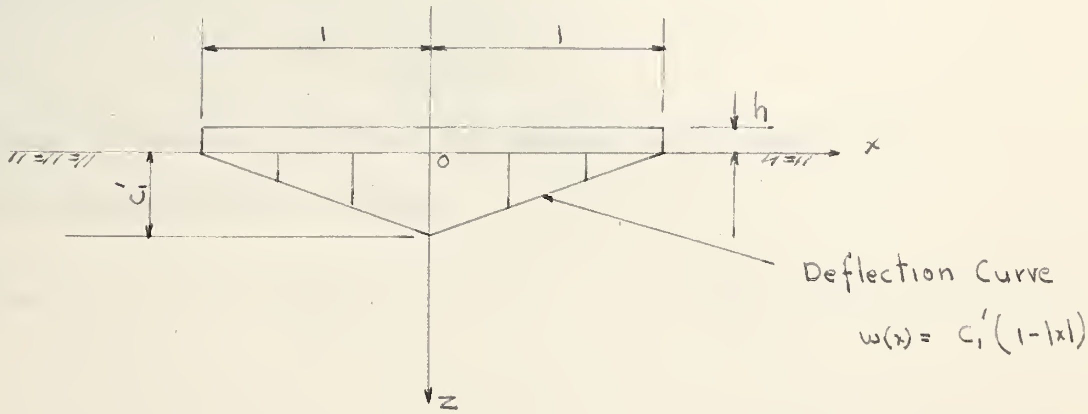

FIG. 11

The boundary conditions for the plate are

$$w = c_1'(1-x) \qquad x>0$$

$$w = c_1'(1+x) \qquad x<0$$

also

$$\frac{dw}{dx} = -c_1' \qquad x>0$$

$$= +c_1' \qquad x<0$$

It has been shown that the slope can be expressed as

$$\frac{1}{c}\frac{dw}{dx} = q(x) = \frac{1}{\pi}\oint_{-1}^{1}\frac{p(\xi)}{\xi - x}\,d\xi.$$

This expression can be transformed into

$$G(\phi) = \frac{1}{\pi}\oint_{0}^{\pi}\frac{P(\theta)\sin\theta}{\cos\theta - \cos\phi}\,d\theta$$

by letting $x = \cos\phi$ and $\xi = \cos\theta$

By expressing $P(\theta)$ in the series form

$$P(\theta)\sin\theta = \sum_{m=0}^{\infty} A_m \cos m\theta$$

and $G(\phi)$ as

$$G(\phi)\sin\phi = \sum_{m=1}^{\infty} A_m \sin m\phi$$

we have shown by equation (57) that the unknown coefficients A_n.
can be determined when $H(\phi)$ is known.

We have that
$$G(\phi) = -c_1 \qquad \cos\phi > 0 \qquad 0<\phi<\frac{\pi}{2}$$

$$= +c_1 \qquad \cos\phi < 0 \qquad \frac{\pi}{2}<\phi<\pi.$$

Therefore $$H(\phi) = G(\phi)\sin\phi \qquad \text{becomes.}$$

$$H(\phi) = -c_1 \sin\phi \qquad 0<\phi<\frac{\pi}{2}.$$

$$= c_1 \sin\phi \qquad \frac{\pi}{2}<\phi<\pi.$$

From (57) we find

$$A_n = -\frac{2c_1}{\pi}\int_0^{\frac{\pi}{2}} \sin\phi \sin n\phi \, d\phi + \frac{2c_1}{\pi}\int_{\frac{\pi}{2}}^{\pi} \sin\phi \sin n\phi \, d\phi \quad \text{---(63)}$$

We may write

$$-\sin n\phi \sin\phi = \frac{1}{2}\left(\cos(n+1)\phi - \cos(n-1)\phi\right)$$

and $$\sin n\phi \sin\phi = -\frac{1}{2}\left(\cos(n+1)\phi - \cos(n-1)\phi\right)$$

Substituting in (63) we have

$$A_n = \frac{c_1}{\pi}\int_0^{\frac{\pi}{2}}\left[\cos(n+1)\phi - \cos(n-1)\phi\right]d\phi - \frac{c_1}{\pi}\int_{\frac{\pi}{2}}^{\pi}\left[\cos(n+1)\phi - \cos(n-1)\phi\right]d\phi.$$

Integrating, the coefficients are given by

$$A_n = \frac{2c_1}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{n+1} - \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right] \quad \text{---(64)}$$

for $n>1$. For $n=1$ we have

$$A_1 = -\frac{2c_1}{\pi}\int_0^{\frac{\pi}{2}} \sin^2\phi \, d\phi + \frac{2c_1}{\pi}\int_{\frac{\pi}{2}}^{\pi} \sin^2\phi \, d\phi$$

which shows that

$$A_1 = 0$$

From the trigonometeric functions we have

$$\sin(n+1)\frac{\pi}{2} = \cos\frac{n\pi}{2} \qquad \text{and} \qquad \sin(n-1)\frac{\pi}{2} = -\cos\frac{n\pi}{2}.$$

therefore (64) becomes

$$A_n = \frac{2C_1}{\pi}\left[\cos\frac{n\pi}{2}\left(\frac{1}{n+1} + \frac{1}{n-1}\right)\right] \qquad n > 1 \qquad \text{---(65)}$$

(65) gives that all the coefficients

$$A_{2n+1} = 0 \qquad n \geq 1$$

and

$$A_{2n} = \frac{2C_1}{\pi}(-1)^n\left[\frac{1}{2n+1} + \frac{1}{2n-1}\right] \qquad n \geq 1 \qquad \text{---(66)}$$

The reaction function can then be expressed as

$$p(\cos\theta)\sin\theta = A_0 + \frac{2C_1}{\pi}\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{2n+1} + \frac{1}{2n-1}\right]\cos 2n\theta \qquad \text{---(67)}$$

To evaluate the series expression of (67) we consider the function

$$f(z) = \frac{2C_1}{\pi}\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{2n+1} + \frac{1}{2n-1}\right]z^{2n} \qquad \text{---(68)}$$

and select only the real part of the answer. In (68) we have that

$$z = e^{i\theta} = \cos\theta + i\sin\theta$$

and

$$z^{2n} = e^{2in\theta} = \cos 2n\theta + i\sin 2n\theta$$

Let
$$f(z) = f_1(z) + f_2(z)$$

and consider each function separately. We have then

$$f_1(z) = \frac{2c_1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{2n+1} \qquad \text{--- (69)}$$

To sum the series given by (69) we multiply both sides by z and differentiate.

$$\frac{d}{dz}\left(z f_1(z)\right) = \frac{2c_1}{\pi} \sum_{n=1}^{\infty} (-1)^n Z^{2n} \qquad \text{--- (70)}$$

The resulting geometeric series can be summed as follows:

$$\frac{d}{dz}\left(z f_1(z)\right) = -\frac{2c_1}{\pi} \frac{z^2}{1+Z^2}$$

To integrate the above expression the l. h. s. is split into partial fractions and we find

$$\frac{d}{dz}\left(z f_1(z)\right) = -\frac{2c_1}{\pi}\left(1 - \frac{1}{1+Z^2}\right)$$

Integrating we have

$$f_1(z) = -\frac{2c_1}{\pi}\left[1 - z^{-1} \arctan z\right] \qquad \text{--- (71)}$$

In a similar manner we find for

$$f_2(z) = \frac{2c_1}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{2n-1}$$

and the resulting expression becomes

$$f_2(z) = -\frac{2c_1}{\pi} z \arctan z \qquad \text{--- (72)}$$

Adding (71) and (72) we have

$$f(z) = -\frac{2c_1}{\pi}\left[1 + (z - z^{-1}) \arctan z\right]$$

$$= -\frac{2c_1}{\pi}\left[1 + (e^{i\theta} - e^{-i\theta}) \arctan e^{i\theta}\right]$$

Recalling that

$$2i \sin\theta = e^{i\theta} - e^{-i\theta}$$

we have on substitution that

$$f(z) = -\frac{2c_1}{\pi}\left[1 + 2i \sin\theta \ \arctan e^{i\theta}\right] \quad ---(73)$$

The $\arctan z$ can be written as

$$\arctan z = \frac{1}{2i} \log \frac{1+iz}{1-iz}.$$

Therefore (73) becomes

$$f(z) = -\frac{2c_1}{\pi}\left[1 + \sin\theta \left\{ \log(1 + i\cos\theta - \sin\theta) - \log(1 - i\cos\theta + \sin\theta)\right\}\right]$$

In complex variable the principal value of the logarithmic function is given as

$$\log z = \log|z| + i \arg z.$$

and thereby the function $f(z)$ can be written as

$$f(z) = -\frac{2c_1}{\pi}\left[1 + \sin\theta \left\{ \frac{1}{2} \log \frac{1-\sin\theta}{1+\sin\theta} + i\left(\arctan \frac{\cos\theta}{1-\sin\theta} - \arctan \frac{(-)\cos\theta}{1+\sin\theta} \right) \right\} \right]$$

which reduces to

$$f(z) = -\frac{2c_1}{\pi}\left[1 + \frac{\sin\theta}{2}\left(\log \frac{1-\sin\theta}{1+\sin\theta} + i\pi \right) \right] \quad -- (74)$$

Selecting only the real part of (74) we have for the reaction function

$$p(\cos\theta)\sin\theta = A_0 - \frac{2c_1}{\pi}\left[1 + \frac{\sin\theta}{2} \log \frac{1-\sin\theta}{1+\sin\theta} \right]$$

and

$$p(\xi) = \frac{A_0 - \frac{2c_1}{\pi}}{\sqrt{1-\xi^2}} - \frac{c_1}{\pi} \log \frac{1-\sqrt{1-\xi^2}}{1+\sqrt{1-\xi^2}}$$

$$p(\xi) = \frac{A_0'}{\sqrt{1-\xi^2}} - \frac{2c_1}{\pi} \log \frac{1-\sqrt{1-\xi^2}}{|\xi|}$$

In terms of the variable x the foundation reaction for linear displacement becomes

$$p(x) = \frac{A_0'}{\sqrt{1-x^2}} - \frac{2c_1}{\pi} \log \frac{1-\sqrt{1-x^2}}{|x|}. \quad -- (75)$$

The determination of the constant A_0' will depend on the contact pressure at the edges of the slab. When the contact pressure is assumed finite at the edges, we have from (75) that A_0' must be zero and therefore the foundation reaction for this case becomes

$$p(x) = \frac{2c_1}{\pi} \log \frac{|x|}{1-\sqrt{1-x^2}} \qquad \text{---(76)}$$

From the slab equation we have that $q(x) = p(x)$

This solution gives an approximation to the concentrated load as shown in figure 12.

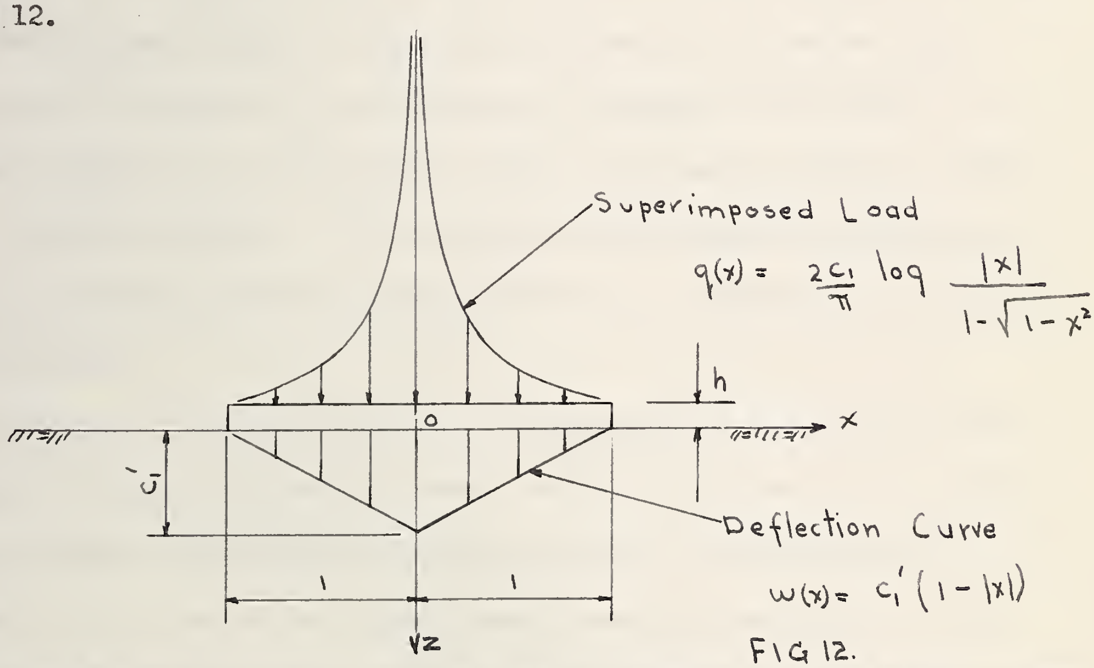

FIG 12.

It is interesting to note that the contact pressure and the superimposed load are equal and opposite and independent of the physical properties of the slab and the foundation for all deflection curves of the cubic order or less. For deflection curves higher than the cubic the superimposed load is dependent on the physical properties of the slab and foundation and is given by the following expression

$$q(x) = D\frac{d^4 w}{dx^4} + p(x).$$

Conclusions

It has been shown that Borowicka's method of analysis is valid only for a special class of problems. The nature of the problem must be such that the deflection function and all its derivatives be continuous. This means that the slope, moment, shear, and load functions cannot have a singularity at any point over the width of the slab. It can be readily seen that such stringent conditions limit the applications of the method in actual practice. To solve the problem which has a discontinuous loading, for example the concentrated load, or discontinuous uniform load on the slab other series expressions for the contact pressure and deflection curve would have to be assumed; or other methods of analysis investigated. An approximate solution for concentrated loads is obtained as suggested under the part "Inverse Problem". This particular method could be extended and a variety of physical problems could be built up by applying the principle of superposition.

The solution of equation (52) for known deflection curves would give the contact pressure distribution beneath actual structures such as locks. A number of these studies made for various types of soils and size of structures could be useful in a preliminary investigation of lock design. For experimental work done in a laboratory, equation (52) would give a means of checking computed and observed contact pressures beneath elastically supported strips. The computed contact pressure being obtained from the observed deflection curve.

In general the solution of the slab resting on elastic foundation would require the solution of the following integral equation

$$w(x) = \frac{2}{\pi N} \int_{-1}^{1} q(x) \left[\log|\xi - x| - \log|\xi| \right] d\xi$$

$$- \frac{2}{\pi N} \int_{-1}^{1} D \frac{d^4 w}{dx^4} \left[\log|\xi - x| - \log|\xi| \right] d\xi. \quad ---(77)$$

Expression (77) is obtained by the coalescence of the slab equation

$$D \frac{d^4 w}{dx^4} = q(x) - p(x)$$

and the expression for the deflection of the foundation

$$w(x) = \frac{2}{\pi N} \int_{-1}^{1} p(\xi) \left[\log|\xi - x| - \log|\xi| \right] d\xi.$$

Another method of obtaining an exact solution to the problem of the discontinuous load would be from a consideration of (52). It has been shown that

$$\frac{1}{c} \frac{dw}{dx} = G(\phi) = \frac{1}{\pi} \int_{0}^{\pi} \frac{P(\theta) \sin\theta \, d\theta}{\cos\theta - \cos\phi} \quad ---(78)$$

We know that the solution of (78) can be obtained when $G(\phi)$ is expressed in the series form

$$G(\phi) = \sum_{m=1}^{\infty} A_m \frac{\sin m\phi}{\sin\phi}$$

and the contact pressure is given as

$$P(\theta) = \sum_{m=0}^{\infty} A_m \frac{\cos m\theta}{\sin\theta}$$

A solution of the foundation reaction can then be obtained from a consideration of the slab equation in which each term is expressed by its corresponding series expansion. That is, the superimposed load on the slab is given by a Fourier series with known coefficients C_m and the fourth derivative of the deflection and the contact pressure are similarly expressed in series form with unknown coefficients A_m. From the resulting expression the equating of coefficients permits a determination of the unknown coefficients A_m to be made in terms of the known coefficients C_m and subsequently a solution to the problem is obtained.

The solution of the continuously loaded strip as given under cases 1, 2, 3, for which $p(x)$ is assumed to be a power series, can be determined with less effort from a consideration of the integral

$$\frac{1}{c}\frac{dw}{dx} = \frac{1}{\pi}\oint_{-1}^{1}\frac{p(\xi)}{\xi - x}\,d\xi \qquad --(79)$$

than tha solution obtained from the equation given by Borowicka as

$$w_0 - w = \frac{2}{\pi N}\int_{-1}^{1} p(\xi)\left[\log|\xi - x| - \log|\xi|\right]d\xi$$

Equation (79) can be placed into a more useful form by writing

$$\frac{1}{c}\frac{dw}{dx} = \frac{1}{\pi}\oint_{-1}^{1}\frac{p(\xi)-p(x)}{\xi - x}\,d\xi + \frac{1}{\pi}p(x)\oint_{-1}^{1}\frac{d\xi}{\xi - x}$$

This expression reduces to

$$\frac{1}{c}\frac{dw}{dx} = \frac{1}{\pi}\oint_{-1}^{1}\frac{p(\xi)-p(x)}{\xi - x}\,d\xi + \frac{1}{\pi}p(x)\left[\log(1+x) + \log(1-x)\right]$$

on evaluation of the second integral.

The solution of the contact pressure for the elastic strip on an elastic foundation as given under cases 1, 2 and 3 can be applied also to indeterminate structures such as culverts. The moments at the corners of the culvert could be determined by using the method of moment distribution.

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